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Circularly Symmetric Static Metric in Three Dimensions and Its Killing Symmetry

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Abstract In this note we consider a general *three*-dimensional circularly symmetric static metric and investigate possible Killing symmetries it possesses.We then summarize the work by addressing Einstein equations and briefly discuss their implications on the energy momentum content of some of the metrics that arise in the process.

Keywords Einstein equations · Symmetries · Killing vectors

1 Introduction

The field equations that depict general relativity are highly non linear in nature [1]. This inherent nonlinear nature of these equations makes it quite a challenging problem to find their exact solutions. On the other hand if we choose energy momentum tensor to define Einstein field equations, then any Lorentzian metric g_{ab} will form a solution of the equations. However, such (arbitrary) solutions are generally of no physical interest as they may not represent physically plausible situations. To overcome the problem in finding physically meaningful solutions, certain restrictions are imposed on the metric g_{ab}. These restriction are generally interpreted in terms of Killing vectors (kvs) **k** along which Lie derivative of the metric g_{ab} is zero, i.e., $\pounds_k g_{ab} = 0$. Physically, existence of kvs in a given spacetime imply existence of conservation laws and are in one-to-one correspondence with continuous

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symmetries of a metric on the manifold. It is well known that if the metric g_{ab} of a given spacetime manifold is flat, it admits maximum $\frac{n(n+1)}{2}$ kvs [1–3]. A lot of work has been done on classification of 4-spacetimes admitting spherical, axial and other symmetry [2]. Of the well known static spherically symmetric spacetimes Minkowski and De-Sitter are maximally symmetric admitting 10 kvs, whereas Einstein, Bertotti-Robinson and Schwarzschild spacetimes admit 7, 6 and 4 kvs respectively. The respective symmetry groups corresponding to maximal symmetry are $SO(1,4)$ and $SO(2,3)$, whereas the minimal symmetry group corresponds to $SO(3) \otimes R$ [2]. In axially symmetric class of 4-spacetime metrics the most general solution of the Einstein equations is given by Kerr/Kerr-Newman that admit *two* kvs only; one temporal and other azimuthal [1].

Apart from others, one of the intriguing features of general theory is existence of black holes which possess both geometrical as well as thermodynamic properties [4, 5]. Whereas thermodynamic properties of the black-holes are concerned they are derived from geometrical degrees of freedom. In their paper Berredo-Peixoto [4] suggest that it is not clear as to how black hole thermodynamics can be affected if one considers a theory of gravity having additional degrees of freedom besides the metric. To be able to approach their investigations Berredo-Peixoto considered three dimensional gravity [6] coupled to a scalar field with special attention to black hole configuration. This study was interesting as it highlighted the importance of three dimensional gravity. An other consideration that supports idea of three dimensional gravity is due to the investigations of Banâdos et al. [7] who first discovered the black hole solution there and discussed various interesting features. Later Clêment and Fabbari [8, 9] discussed solutions in $2+1$ dimensions for minimally coupled massless scalar field with non-trivial cosmological constant. More recently, Gracia et al. [10] and Sousa and Maluf [11] developed three dimensional theory of gravity describing black hole solution. Berredo-Peixoto investigated circularly symmetric static solutions of Einstein field equations in three dimensions assuming that there do not exist black holes but naked singularity [4]. In this work they present a numerical solution of the field equations containing a naked singularity that is consistent with no-hair theorem and no-black hole theorem in three dimensions [12].

In the light of above discussion it is clear that three dimensional gravity theory may be significant for the role it may play in explaining some of the very crucial aspects of the 3-spacetime physics. With this point in mind it appears to be an interesting problem to understand three dimensional spacetime geometries by their symmetries such as kvs etc. In this spirit we consider a generalized version of the circularly symmetric metric considered by Berredo-Peixoto [4] and give a classification of its Killing symmetries. To do so we follow techniques developed earlier by Bokhari et al. [13, 14]. In the process of this classification some cases arise in which metrics obtained admit similar Killing vectors. We however consider only those cases for which distinct kvs are found. The plan of this note is as follows: In the next section we briefly derive Killing vectors of the metric which is a generalization of the metric considered in [4] and give algebra satisfied by their symmetry generators. In the last section we summarize results by addressing Einstein field equations to some metrics found in process.

2 Brief Derivation of Killing Symmetry

The circularly symmetric metric given by Berredo-Peixoto [4] can be written generally in the form,

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\theta^2. \tag{2.1}$$

For the metric, (2.1), the Killing's equations $\pounds_k g_{ab} = 0$ form a system [1]

$$x_{ab} : g_{ab,c}k^c + g_{ac}k^c_{,b} + g_{ac}k^b_{,a} = 0 \tag{2.2}$$

of *six* coupled partial differential equations in three unknowns k^a $(a = 0, \dots, 2)$ whose solutions can be written in the form $\overline{k} = k^a \frac{\partial}{\partial x^a}$. To start classification of kvs we first consider x_{11} and integrate it over r to get,

$$k^1 = A(t, \theta)e^{-\frac{\lambda(r)}{2}}, \tag{2.3}$$

where $A(t, \theta)$ is a function of integration. From (2.3) two possibilities arise, namely, (I): $A(t, \theta) = 0$ and (II): $A(t, \theta) \neq 0$. We consider these two cases one by one and give brief derivation of kvs.

Case I

In this case four of the Killing equations, x_{oo}, x_{o1}, x_{12} and x_{22}, give $k^o = B(\theta)$ and $k^2 = C(t)$. Differentiating x_{o2} with respect to θ and t respectively, with values of k^o and k^2 in mind, instantly imply that B and C are linear in θ and t, i.e., $B = c_o + c_1\theta$ and $C = c_2 + c_3 t$, with c_o to c_3 some integration constants. To require consistency, we then use the resulting expressions of k^o and k^2 in x_{o2} to get,

$$e^{\nu}c_1 - r^2 c_3 = 0. \tag{2.4}$$

From (2.4) two cases arise, namely, (a): $e^{\nu(r)} = r^2$; $c_1 = c_3$ and (b): $e^{\nu(r)} \neq r^2$; $c_1 = 0 = c_3$. In case (a) it turns out that the system (2.2) yields *three* kvs $\overline{k} = (c_o + c_1\theta)\frac{\partial}{\partial t} + (c_2 + c_1 t)\frac{\partial}{\partial \theta}$. In case (b) the system (2.2) admits minimal (circular) symmetry that is given by $\overline{k} = c_o\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial \theta}$. The symmetry generators associated with above kvs in both cases are $X_o = \frac{\partial}{\partial t}$, $X_1 = \theta\frac{\partial}{\partial t} + t\frac{\partial}{\partial \theta}$, $X_2 = \frac{\partial}{\partial \theta}$, and $X_o = \frac{\partial}{\partial t}$, $X_2 = \frac{\partial}{\partial \theta}$ respectively. Further, whereas generators in case (Ia) satisfy $[X_1, X_2] = X_2$, $[X_2, X_1] = X_o$ and $[X_o, X_2] = 0$, in case (Ib) they commute ($[X_i, X_j] = 0$) with each other for all i.

Case II

In this case $A(t, \theta) \neq 0$, thus k^1 is defined by (2.3). To find other components of kv, we first integrate x_{oo} and x_{22} with respect to t and θ respectively and get,

$$k^o = -\frac{\nu(r)' e^{-\frac{\lambda(r)}{2}}}{2}\int A(t, \theta)dt + E(r, \theta), \tag{2.5}$$

$$k^2 = -\frac{e^{\frac{\lambda(r)}{2}}}{r}\int A(t, \theta)d\theta + D(t, r), \tag{2.6}$$

where $E(r, \theta)$ and $D(t, r)$ are some functions of integration to be determined. For this purpose we first differentiate x_{22} with respect to θ and then use the resulting expression in x_{12} to get a second order differential equation

$$k^2_{,\theta\theta} - re^{-\lambda(r)}k^{2\prime} = 0, \tag{2.7}$$

where ($'$) is used to represents differentiation with respect to the radial coordinate. Now using value of k^2 in (2.7) and requiring consistency there gives,

$$\frac{A(t,\theta)_{,\theta\theta}}{A(t,\theta)} = -e^{-\lambda(r)}\left(1+\frac{r\lambda(r)'}{2}\right) = \alpha, \tag{2.8}$$

where α is a separation constant and can be (a): $\alpha = k^2 > 0$, (b): $\alpha = -k^2 < 0$ and (c): $\alpha = 0$. We give brief calculations of kvs in the first case only. Results in cases (IIb) and (IIc) will be mentioned with out giving detail.

Case IIa

In this case (2.8) gives rise to two equations, namely,

$$A(t,\theta)_{,\theta\theta} - k^2 A(t,\theta) = 0, \tag{2.9}$$

$$e^{-\lambda(r)}\left(1+\frac{r\lambda(r)'}{2}\right) = -k^2. \tag{2.10}$$

Equation (2.9) is a second order partial differential equation that can be easily solved to get

$$A(t,\theta) = A_1(t)\cosh(k\theta) + A_2(t)\sinh(k\theta). \tag{2.11}$$

Using (2.11) in (2.3) and (2.5)–(2.6), with (2.10) in mind and requiring consistency, one finds that $D(t,r)$ can only be a function, $F(t)$, of time coordinate. Using this fact and (2.11), the three components of the kv become,

$$\begin{aligned}
k^o &= -\frac{\nu(r)'e^{-\lambda(r)/2}}{2}\left[\cosh(k\theta)\int A_1(t)dt + \sinh(k\theta)\int A_2(t)dt\right] + E(r,\theta),\\
k^1 &= e^{-\lambda(r)/2}[A_1(t)\cosh(k\theta) + A_2(t)\sinh(k\theta)],\\
k^2 &= -\frac{e^{\lambda(r)/2}}{r}[A_1(t)\sinh(k\theta) + A_2(t)\cosh(k\theta)] + F(t).
\end{aligned} \tag{2.12}$$

The next step now is to determine the unknown functions appearing in (2.12). This can be achieved by addressing the Killing equations (x_{ab}) with (2.12) in mind and solving the resulting differential constraints that arise in the process of checking consistency. Following these steps it is easily found that the system (x_{ab}) of Killing's equation admits *six* kvs given by:

$$\begin{aligned}
k^o &= -\frac{\nu(r)'e^{-\lambda(r)/2}}{2p}[\cosh(k\theta)(c_1\sinh(pt) + c_2\cosh(pt)\\
&\quad + \sin(k\theta)(c_3\sinh\sqrt{\beta}t + c_4\cosh(pt)] + c_o,\\
k^1 &= e^{-\lambda(r)/2}[\cosh(k\theta)(c_1\cosh(pt) + c_2\sinh(pt)\\
&\quad + \sinh(k\theta)(c_3\cosh(pt) + c_4\sinh(pt)],\\
k^2 &= -\frac{e^{\lambda(r)/2}}{rk}[\sinh(k\theta)(c_1\cosh(pt) + c_2\sinh(pt)\\
&\quad + \cosh(k\theta)(c_3\cosh(pt) + c_4\sinh(pt)] + c_5,
\end{aligned} \tag{2.13}$$

where $e^{\nu(r)} = (\frac{p^2}{k^2}r^2 - m)$ and is related to $\lambda(r)$ by the differential constraints $e^{\nu}(-2\nu'' + \nu'\lambda') = 4p^2e^{\lambda}$ and $e^{\lambda}(1 + r\lambda'/2) = -k^2$.

In (IIb) the number of kvs are same as in the (IIa) with the only difference that the hyperbolic functions there get replaced with trigonometric ones.

Case IIc

In this case $e^{\nu} = br^m$ and $e^{\lambda} = \beta r^{-2}$. Here there can be two possibilities for m, i.e., (x): $m \neq 0$ and (y): $m = 0$. We first consider case (x). Using values of e^{ν} and e^{λ} in (2.1) and then solving the resulting Killing's equations one finds that given 3-spacetime admits three Kvs given by,

$$\begin{aligned} k^0 &= c_o - c_1\frac{m}{2}t, \\ k^1 &= c_1 r, \\ k^2 &= c_2 - c_1\theta. \end{aligned} \tag{2.14}$$

The three symmetry generators constructed from these kvs are: $X_o = \frac{\partial}{\partial t}$, $X_1 = \frac{-mt}{2}\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \theta\frac{\partial}{\partial\theta}$ and $X_2 = \theta\frac{\partial}{\partial\theta}$ whose algebra is determined by the commutation relations $[X_0, X_1] = \frac{-m}{2}X_0$, $[X_0, X_2] = 0$ and $[X_1, X_2] = X_2$, where $X_o = \frac{\partial}{\partial t}$, $X_1 = -\frac{mt}{2}\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} - \theta\frac{\partial}{\partial\theta}$, $X_2 = \frac{\partial}{\partial\theta}$ respectively. In case (y), $e^{\nu} = 1$ (with b absorbed into the definition of e^{ν}) whereas value of e^{λ} remains the same as in case (x). Using these values in (2.1) and solving them as before ones finds that the 3-spacetime admits 4 kvs given by,

$$\begin{aligned} k^0 &= c_o, \\ k^1 &= (c_1 + c_2\theta)r, \\ k^2 &= -c_1\theta - \frac{1}{2}\left(\theta^2 - \frac{q^2}{r^2}\right)c_2 + c_3. \end{aligned} \tag{2.15}$$

The symmetry generators in this case are given by: $X_o = \frac{\partial}{\partial t}$, $X_1 = r\frac{\partial}{\partial t} - \theta\frac{\partial}{\partial\theta}$, $X_2 = r\theta\frac{\partial}{\partial r} - \frac{\theta^2}{2}\frac{\partial}{\partial\theta} + \frac{q^2}{2r^2}\frac{\partial}{\partial\theta}$ and $X_3 = \frac{\partial}{\partial\theta}$. The algebra of these generators is determined by the combination relations given: $[X_0, X_i] = 0$, $[X_1, X_2] = -X_2$, $[X_1, X_3] = X_3$ and $[X_2, X_3] = -X_1$ respectively.

Case III

This case corresponds to asymptotic form of the metric (2.1) in the limit as r approaches infinity,

$$ds^2 = -dt^2 + dr^2 + r^2d\theta^2. \tag{2.16}$$

It is a maximally symmetric metric in 3-dimensions and therefore admits *six* Kvs given by:

$$\begin{aligned} k^0 &= c_o + r(c_1\cos\theta + c_2\sin\theta), \\ k^1 &= t(c_1\cos\theta + c_2\sin\theta) + (c_3\cos\theta + c_4\sin\theta), \\ k^3 &= \frac{1}{r}[t(-c_1\sin\theta + c_2\cos\theta) + (-c_3\sin\theta + c_4\cos\theta)] + c_5. \end{aligned} \tag{2.17}$$

The six symmetry generators corresponding to above kvs form a $G\langle 6\rangle$ whose algebra is determined by the commutation relations $[X_0, X_1] = X_3$, $[X_0, X_2] = X_4$, $[X_1, X_2] = X_5$, $[X_1, X_3] = -X_0$, $[X_1, X_5] = X_2$, $[X_2, X_4] = -X_0$, $[X_2, X_5] = X_1$, $[X_3, X_5] = X_4$, $[X_4, X_5] = -X_3$, with all the remaining generators commuting with each other, i.e., $[X_i, X_i] = 0$.

3 Summary and Discussion

Considering a generalized version of the circularly symmetric static metric given by Berredo-Peixoto [4], we have derived its Killing vectors in some cases. The maximal Killing symmetry of this metric corresponds to two cases, namely, (IIa) and (III). Whereas case (III) is static flat 3-spacetime, in case (IIa) the g_{oo} component of the metric is determined explicitly with g_{11} subject to a differential constraint. From (IIc) two cases (IIcx) and (IIcy) arise with each giving 3 and 4 kvs respectively. In (IIcx) whereas all the Ricci tensor components are non-zero, the R_{00} is zero in the later case. Consequently, symmetry of the Ricci tensor (also called Ricci collineations [2, 3]) in (IIcx) are similar to the kvs and arbitrary in temporal direction in (IIcy). Case (III) is Ricci flat (all Ricci tensor components zero identically) and therefore Ricci collineations are arbitrary in all directions. To study some of the metrics further, we use Einstein field equations. In case (Ia) the Einstein equations determine the three energy momentum tensor components given by $T_0^0 = -e^{-\lambda(r)}\lambda'/2r = T_2^2$ and $T_1^1 = e^{-\lambda(r)}/r^2$. From here it is easy to note that the energy content of the spacetime is positive provided λ' is negative. To determine λ explicitly we restrict energy momentum tensor to zero trace condition. Solving the resulting equation immediately yields $e^{\lambda} = r$ giving form of the 3-spacetime metric as $ds^2 = -r^2dt^2 + rdr^2 + r^2d\theta^2$. Cases (Ib) and (IIa) are arbitrary and therefore not considered. In case (IIcx) the Einstein equations give $T_0^0 = \frac{1}{\beta} = \frac{2}{m}T_2^2 = \frac{4}{m^2}T_3^3$. In case (IIc$y$) the only non-zero component of the energy momentum tensor is $T_0^0 = \frac{1}{\beta}$ with all other components zero. In case (III) the metric is given by $ds^2 = -dt^2 + dr^2 + r^2d\theta^2$. This is a flat metric and has no energy momentum content as expected. To understand gravitational and other relativistic effects in 3-spacetime geometries, it may be worth studying them in more detail.

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